WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4701. Sketch the graph $y = \mathrm{f}(x)$ and $y = \frac{1}{4}x$. Show that there are precisely four intersections. You don't need to solve to find them.

4702. It's easier if you scale the triangles to have side length 1. Work out the angles and so the side lengths in the six right-angled triangles around the outside. Find the area of these. Then subtract the area of three of them from the area of one of the equilateral triangles.

4703. Use a double-angle formula, and parts.

4704. Rotate the scenario so that the reaction forces are $-R\mathbf{i}$, $-R\mathbf{j}$ and $-R\mathbf{k}$.

4705. Expand with compound-angle formulae.

4706. Write the squared distance between (x, y) and $(1, 1)$ in terms of x, and optimise using calculus. You will also need to consider the extreme points of the curve.

4707. This is a rephrasing of the AM-GM inequality:

$$\tfrac{1}{2}(a + b) \geq \sqrt{ab}.$$

It is proved by starting with $(a - b)^2 \geq 0$.

4708. Call the integral (or perhaps $\frac{1}{41}$ of it) I, and then integrate twice by parts. Use the tabular method for ease, converting the line-zero integral (as a product) into the line-two integral (as a product). This can be written in terms of I. Then rearrange to make I the subject.

4709. A stationary point on the x axis is a double root. So, the quartic must be expressible as

$$\begin{aligned} &4x^4 + 4x^3 + kx^2 - 2x + 1 \\ \equiv\ &4(x - a)^2(x - b)^2. \end{aligned}$$

4710. The entire map has rotational symmetry around the origin. So, the shortest pass must pass through O. Call the equation of the path $y = kx$. Use the discriminant to find the value of k for which this has only one intersection with $y = (x - 1)(x - 4)$.

4711. Use the generalised binomial expansion.

4712. Divide both sides by e^x. Use an identity to convert the sines into cosines, and then factorise. Sketch the graphs of $y = \cos x$ and $y = e^{-x}$, and explain visually why the roots must appear in pairs.

4713. Use the substitution $3x + 1 = \tan\theta$.

4714. The first equation is quadratic in y.

4715. Assume, for a contradiction, that there are finitely many prime numbers. Call them

$$p_1 < p_2 < ... < p_n.$$

Let $P = p_1 p_2 ... p_n$. Show that $P + 1$ is prime, and explain the contradiction.

4716. (a) Set up $\mathrm{g}''(x) = \mathrm{h}''(x)$ and integrate twice.
(b) Differentiate using the quotient rule, and set the numerator to zero. Let $\mathrm{g}(x) = \mathrm{h}(x) + k$ and substitute in, eliminating the function g. At the end, use the fact that h is quartic to justify the presence of at least one root.

4717. (a) Use Pythagoras and the given identity.
(b) Perform similar procedures as in part (a), for $|BX|$ and $|CX|$. In each case (they can be done together with a $\mp$), you'll need to express the trig in harmonic form in order to reuse the given half-angle identity. At the end, expand with a compound-angle formula.

4718. Let $X = x - a$ and $Y = y - b$, so the graphs are
① $X^2Y^2 = 1$,
② $X^2 + Y^2 = 2$.

Analyse these fully, then translate.

4719. Expand the RHS with a compound-angle formula, then use the small-angle approximations.

4720. Consider the positive quadrant only. Then $|xy|$ is simply xy. Write each of x and y in terms of θ. Integrate xy with respect to θ (this is analogous to "adding up" all the values of xy), then divide by the length of the arc (this is analogous to dividing by the number of items).

4721. Find the equation of a generic normal at (p, p^2), and its y intercept. Then express the relevant area as that of a rectangle plus that of a triangle, minus that under the curve.

4722. Find $\frac{dy}{dx}$ in terms of t, simplifying fully. Evaluate at the relevant t value to determine the equation of the tangent at P. Substitute for x and y. Use an identity to produce a cubic in $\sin t$. Take out a double factor (from the point of tangency), and consider the single factor that remains.

4723. Factorise. Then use the fact that a product is non-positive iff exactly one of its factors is non-positive.

4724. It is possible: find an example.

4725. Write $\sin^2 x \equiv 1 - \cos^2 x$, and then integrate by inspection using the reverse chain rule.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4726. Rearrange the first equation and then sub into the second. Use N-R to solve the resulting equation. The curves are symmetrical in $y = x$, so you only need find two points of intersection.

4727. Find $f'(x)$, which is a quartic. You need to show that it is positive everywhere. So, differentiate again, setting $f''(x) = 0$. Solve this, showing that there is only one x for which $f'(x)$ is stationary. Show that $f'(x) > 0$ at this value.

4728. Multiply up by the denominators. Then equate coefficients of x^2 and equate constant terms. You should reach a contradiction.

4729. The rope is smooth, so the tension is the same everywhere. By symmetry, therefore, the lines of action must lie along the angle bisectors. Let the half-angles at A, B, C, D be $\alpha, \beta, \gamma, \delta$. You know

$$\alpha + \beta + \gamma + \delta = 180°.$$

Consider two triangles, those formed of A, B and the intersection of their angle bisectors, and C, D and the intersection of their angle bisectors.

4730. (a) Square both equations, and use two identities to write y^2 in terms of x^2.

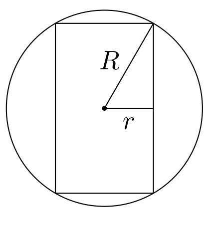
(b) At a point of self-intersection, two different values of the parameter produce the same (x, y). Call these s, t. Set up $\cos s = \cos t$, and solve, noting that $s \neq t$.

4731. (a) Use product and chain rules. Then substitute into the LHS of the DE. Simplify fully; the value of k should become obvious.

(b) Consider the behaviour of $P = e^{-2x}\sin 3x$, as compared to $P = \sin 3x$. The factor e^{-2x} can be thought of as an amplitude.

4732. Let P be a point of tangency for $x > 0$. The x intercept k of the tangent line is minimised if P is the point of inflection of the curve.

4733. Consider the boundary case, in which the cylinder has maximal volume. Its ends (the circumferences of) must touch the surface of the sphere. In cross-section, the scenario is



Write the height h and then the volume V of the cylinder in terms of the radius r. Then set the derivative $\frac{dV}{dr}$ to zero, and optimise.

4734. Differentiate to show that

$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}+1}{x^2\sqrt{x^2-1}}.$$

Set this to $\frac{3}{10}$. Rearrange the equation, squaring it to give a cubic in x^2. Solve this with a polynomial solver.

4735. Since the graph $y = h(x)$ is symmetrical around $x = 0$, the mean of the roots is 0. Hence, we can express them, in terms of a constant k, as $\{-3k, -k, k, 3k\}$.

4736. It's easier using a general x rather than $x = 1/2$ to begin with. Consider $\sin\big(2\arctan x\big)$. Using the sine double-angle formula, this is

$$2\sin(\arctan x)\cos(\arctan x).$$

To simplify $\sin(\arctan x)$, let $x = \tan y$ and then manipulate the expression with identities. You're looking for something with no y's or trig functions in it. Do the same for $\cos(\arctan x)$, then sub in $x = 1/2$ and simplify.

4737. Find the gradients of the reaction forces in terms of m_1 and m_2, and resolve them into components. Set up horizontal and vertical equations and solve for R_1, R_2.

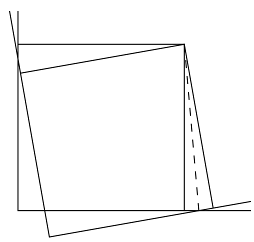
— Alternative Method —

Use a triangle of forces.

4738. Separate the variable to set up two integrals, one with respect to x and one with respect to y. For the x integral, take out a factor of $\sqrt{x^2}$ and thereby integrate by inspection (reverse chain rule). For the y integral, use a double-angle formula.

4739. Use a double-angle formula and a compound angle formula to expand and simplify. Then factorise.

4740. Use the following diagram:



The dashed line bisects the angle of rotation θ. Use the angle $\frac{1}{2}\theta$ to find approximate expressions for the lengths of the perpendicular sides of the eight triangles. Use the small-angle approximation $\tan\theta \approx \theta$ to simplify as you go.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4741. The restricted possibility space consists of all strictly increasing sequences of four scores. There is exactly one increasing sequence for every set of four scores. Therefore, the restricted possibility space contains ${}^6\mathrm{C}_4 = 15$ outcomes.

4742. Set up a general equation of the trajectory with initial speed u and angle of projection θ. Use the second Pythagorean trig identity to rewrite it as a quadratic in $\tan\theta$. On the parabola of safety, the discriminant of this quadratic is zero.

4743. (a) Using the factorial definition of ${}^n\mathrm{C}_r$, simplify the equation to a quadratic in n. Then use the quadratic formula.
(b) For both n and k to be natural numbers, the discriminant must be a perfect square.

4744. Use the substitution $u = \sqrt{x} + 1$.

4745. Show that the shaded area is given by

$$A = 4x\sqrt{2 - x^2} - 4x.$$

Then set the derivative $\frac{dA}{dx}$ to zero and solve a quadratic in x^2.

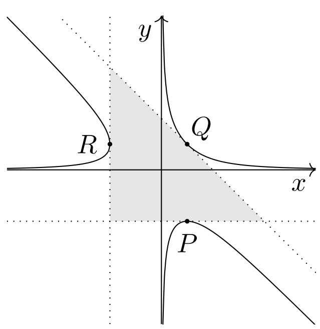
4746. To use the expansion $a^3 - b^3 \equiv (a-b)\big(a^2+ab+b^2\big)$, multiply top and bottom by

$$(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{1}{3}}.$$

The rest is algebra.

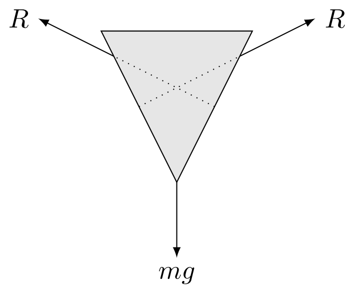
4747. Find the derivatives, simplifying fully at each stage. A problem like this isn't hard as long as you keep on top of the algebra. You should be able to spot the pattern by the second derivative. Substitute into the LHS of the DE, taking out a factor of e^x as you go.

4748. The scenario is



4749. Consider $x \geq 0$ and $x < 0$ separately. In each case, solve a quadratic inequality without mod signs in it, restricting the solution to the relevant domain. Then combine the two solution sets, taking their union.

4750. Find the gradient of the curve $y = \ln(x^2)$ at $x = 1$. Then, since the funnel is smooth, the force applied to the plug must act perpendicularly to this, i.e. normal to the curve. The force diagram for the plug is as follows:



The wording of this question, "per centimetre of contact", means that you need to consider the *total* force (sum of magnitudes of vectors) rather than the *resultant* force (sum of vectors).

4751. (a) Consider the parity of $x + y$.
(b) To end up at $(3, 0)$, there must be at least three $\mathbf{i}$ steps. The remaining two can be $\pm\mathbf{i}$ or $\pm\mathbf{j}$. This gives two possible types of route to $(3, 0)$:
① $\{\mathbf{i}, \mathbf{i}, \mathbf{i}, \mathbf{i}, -\mathbf{i}\}$,
② $\{\mathbf{i}, \mathbf{i}, \mathbf{i}, \mathbf{j}, -\mathbf{j}\}$.

4752. Use a tan double-angle formula and the second Pythagorean identity to write the equation as a quadratic in $\tan^2\psi$. Factorise and solve.

4753. Call the quartic $y = \mathrm{g}(x)$. You know that $\mathrm{g}''(x)$ is quadratic. It is zero at $x = p, q$, so must be a scalar multiple of $ax^2 + bx + c$:

$$\mathrm{g}''(x) = kax^2 + kbx + kc.$$

Integrate this twice, using the various given facts to work out the two constants of integration and then k.

4754. (a) Set $\mathrm{f}'(x)$ to 1, and show that the resulting equation has no roots other than $x = 0$.
(b) Show that $\mathrm{f}''(x)$ is positive everywhere, and therefore that that $\mathrm{f}'(x)$ is increasing. From this fact, find your way to

$$x_{n+2} - x_{n+1} > x_{n+1} - x_n.$$

4755. Solve to find the t values of the y intercepts. Find $\dot{x}$. Then set up the parametric integration formula. It's easiest if you double the area in the positive quadrant. Expand the polynomial to integrate.

4756. The event C is only relevant in that its probability is larger than that of A and B. So, both A and B can be subsets of C. The minimum and maximum, therefore, are determined solely by the numerical values of $\mathbb{P}(A)$ and $\mathbb{P}(B)$.

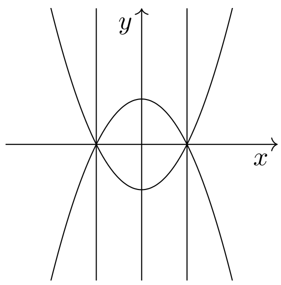
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4757. For the fraction to be less than 1, you require one of the following, where N is the numerator and D the denominator:

- N positive, D negative,
- D positive, N negative,
- N and D both positive, with $N < D$,
- N and D both negative, with $N > D$.

The boundary equation is $y - x^2 + 1 = y + x^2 - 1$, which gives $x = \pm 1$. This is the boundary at $N = D$. There are also possible boundaries at $N = 0$ and $D = 0$. These are $y = x^2 - 1$ and $y = 1 - x^2$. So, the only locations at which there can be boundaries are



4758. Let p be the time of release. Find, in terms of k and p, the initial position and velocity with which the particle becomes a projectile. Set up vertical and then horizontal *suvat*s to find the landing position in terms of k and p. Consider this in harmonic form.

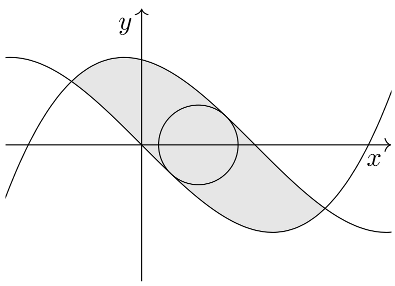
4759. (a) The relevant identity is

$$\begin{aligned}&3x^2 + 2xy + 3y^2\\ \equiv\ &a(x+y)^2 + b(x-y)^2\\ \equiv\ &(a+b)x^2 + (2a-2b)xy + (a+b)y^2.\end{aligned}$$

(b) Define new variables $X = x + y$ and $Y = x - y$.

4760. (a) Set up the equation for intersections. It is a quadratic. Set the discriminant to zero.

(b) The second curve is a translation of the first by vector $\frac{1}{2}\mathbf{i}$. So, the problem has rotational symmetry around the point $C : (1/4, 0)$. Hence, this must be the centre of the largest circle.



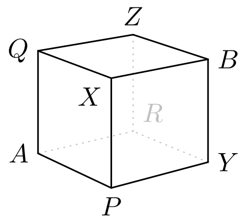
The radius is greatest if the circle is tangent to the curve. So, you need to find the normal to the curve $y = x^3 - x$ which passes through C.

4761. This is just about keeping track of the algebra. Keep things organised: divide and conquer. Set up and simplify separate expressions for

① $2s$ and so s,
② $(s-a)$, $(s-b)$ and $(s-c)$,
③ $s(s-a)(s-b)(s-c)$, and finally
④ A.

4762. Square the boundary equation and substitute in the equation of the circle to produce a quartic in x^4. Use numerical methods to find the roots of this curve and thereby to factorise it. Show that its four roots are all double roots.

4763. There are three vertices which share an edge with A. Call these P, Q, R. Then call the remaining vertices X, Y, Z.



Show that there are six symmetrical choices for the first two edges. Choose $A \to P \to X$. Show, by listing them systematically, that there are three choices from this point.

4764. For $y = k$ to intersect the curve three times, the following equation must have three distinct roots:

$$\begin{aligned}&\frac{1}{x} - \frac{1}{x^3} - k = 0\\ \Longrightarrow\ &kx^3 - x^2 + 1 = 0.\end{aligned}$$

For a cubic to have three distinct roots, it must have stationary values which are +ve and −ve.

4765. (a) Square both sides, then equate coefficients of rational and irrational terms. Then, you can solve a pair of simultaneous equations.

(b) Use the identity $\cos 2\theta = 1 - 2\sin^2\theta$, and then the result from part (a).

4766. Set up a definite integral, and use a trigonometric substitution to evaluate it. Write the new limits in terms of a and follow the equation through.

4767. The equation $\mathrm{h}''(x) = 0$ has exactly two roots. Call them a and b. Assume, for a contradiction, that both are roots of odd multiplicities m and n. Taking out factors of $(x-a)^m$ and $(x-b)^n$, the equation $\mathrm{h}''(x) = 0$ can be expressed as

$$(x-a)^m(x-b)^n\,\mathrm{p}(x) = 0,$$

where $\mathrm{p}(x)$ has no factors of $(x-a)$ or $(x-b)$.

Consider the fact that a polynomial of odd degree must have a root.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4768. Assume, for a contradiction, that the cube root of 5 is rational, and can be written as p/q, where p and q are integers. This gives

$$5q^3 = p^3.$$

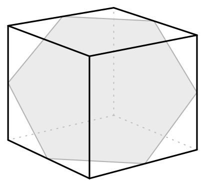
Consider the number of factors of 5 on each side.

4769. Do this by sketching. Solve the first equation for $(x - y)$ to sketch it.

4770. Substitute b into the equation. Divide through by b, and use the discriminant.

4771. Use the cosine rule on $\triangle ABC$ to find an expression for $\cos B$ in terms of a, b, c. Then use the cosine rule on $\triangle ABM$.

4772. The symmetry of the cube and such a regular hexagon dictates that the vertices of the hexagon must be at the midpoints of edges of the cube.



4773. (a) The indefinite integral of g is G. So,

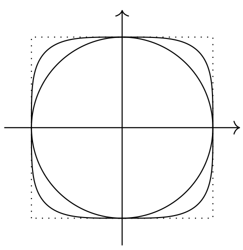
$$\int_0^a \mathrm{g}(x)\,dx = \Big[\mathrm{G}(x)\Big]_0^a = ...$$

(b) Integrate by inspection.

4774. Address the factors one by one:

① For the quadratic, use the discriminant.
② Differentiate and show that both stationary values of the cubic are negative.
③ Consider the fact that both powers are even.

4775. The curves $n = 1$ and $n = 2$ are as follows:



4776. (a) Square y and write in terms of $\cos t$.

(b) Consider the lobe $x \geq 0$. Set up the standard parametric formula for integration. Then use a double-angle identity and inspection.

4777. Consider the behaviour of $\sin(\ln x)$ on the domain $(0, 1]$. Show that $y = \sin(\ln x)$ and $y = x$ intersect infinitely many times over this domain.

4778. From the information given, the functions are

$$\mathrm{f}(x) = (x - a)(x - b),$$
$$\mathrm{g}(x) = (x - b)(x - c).$$

Set up and solve the equation $\mathrm{f}(x) + \mathrm{g}(x) = 0$. You should get two roots, given in terms of a, b, c. You need to show that these are distinct. Assume, for a contradiction, that they are the same and sub $b = ar$ and $c = ar^2$.

4779. Sketch graphs of the LHS and RHS.

4780. Set up or quote the equation of the trajectory

$$y = x\tan\theta - \frac{gx^2}{2u^2}(\tan^2\theta + 1).$$

Then solve this as a quadratic in $\tan\theta$.

4781. Solve as a quadratic in y. This gives a a pair of distinct curves, each of which is symmetrical in the y axis. Find the y intercepts. Then, consider the behaviour as $x \to \pm\infty$.

4782. Here, the gradient of a face is the steepest gradient of a face, which is the gradient of the median. The relevant gradient triangles share the same height (the height of the tetrahedron), so you don't need to calculate it. Consider the ratio of lengths on the base.

4783. Use a double-angle formula, and then factorise the LHS. This will give the equations of each of the branches. Solve them simultaneously.

4784. Use the following identity:

$$(x - y)\big(x^3 + x^2y + xy^2 + y^3\big) \equiv x^4 - y^4.$$

Multiply top and bottom by a version of the factor

$$\big(x^3 + x^2y + xy^2 + y^3\big).$$

4785. Find the derivatives using the product and chain rules. Then set up an identity (the DE must hold for all values of x) and equate coefficients.

4786. Use a numerical method (Newton-Raphson is best) to find a root. Take this out. You should be able to factorise what remains.

4787. The upward force exerted by the pulley on the string is $T_1 + T_2$. So, this is the downwards force exerted by the string on the pulley. If the system moves, then friction is at maximal. The tension T_1 is necessarily larger than the tension T_2, and the difference, we are told, is

$$T_1 - T_2 = \mu(T_1 + T_2).$$

Combine this with the two equations of motion and solve for the acceleration.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

4788. Multiply by x^3 to get a cubic in x^9.

4789. Expand using compound-angle formulae. Combine the expansion, cancelling four of the terms with a Pythagorean identity. Use a double-angle formula on what remains.

4790. Since the quartic has two local minima on the x axis, it has two double roots. So, you must be able to write

$$\begin{aligned} &36x^4 + 12x^3 - 11x^2 - 2x + 1 \\ \equiv\ &36(x-a)^2(x-b)^2. \end{aligned}$$

Equate coefficients.

4791. Let $u = x + y$.

4792. Consider the variables $x + y$ and $x - y$.

4793. Rewrite the algebraic fraction as

$$1 + \frac{1}{x(x+2)}.$$

Work out the range of the denominator, then of the fraction, then of the function.

4794. (a) The boundary gradients are $m = 1$ and $m = 2$.
(b) Set up the conditional probability formula:

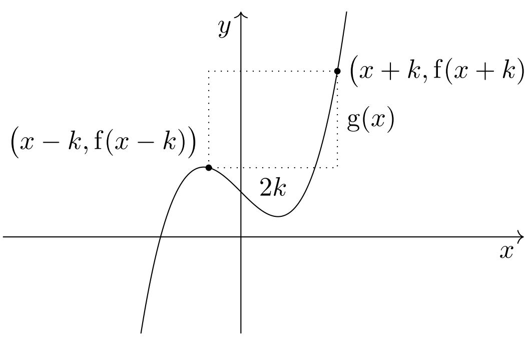
$$\begin{aligned} &\mathbb{P}(\text{steeper than } y = x \mid \text{+ve gradient}) \\ =\ &\frac{\mathbb{P}(M > 1)}{\mathbb{P}(M > 0)}. \end{aligned}$$

4795. Let $x = \sec\theta$.

4796. The factorisation is $(x^2 + ...)(1 - ...)$

4797. Consider the curve $y = \cos^2 x - 2\cos x$. Find SPs and sketch the curve carefully. Then find k such that the line $y = -k$ intersects this curve exactly once.

4798. Show that $\mathrm{g}(x)$ is a quadratic. Then consider the respective symmetries of cubics and quadratics. The relevant graph is



4799. If the tension is too low, then equilibrium will be broken: the upper core will sink downwards, thereby separating the two lower cores. If the tension is high, then the two lower cores will be pressed against one another. In the boundary case between these regimes, the cores are as depicted, but the reaction force between the two lower cores is zero.

4800. Let $u = b + e^{kx}$. Simplify the integrand to the form ${}^{\mathrm{f}(u)}/_{\mathrm{g}(u)}$, where f is linear and g is quadratic. Then write the integrand in partial fractions.

——— End of 48th Hundred ———

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK
WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP